Supplementary File 1: Proof supplements for this study.

We include here all the omitted proofs.

A - Basic properties of the axiom system

These are just results in normal modal logic; we include the proofs for reference.

Lemma 4.4. *The following hold in* A.

1. *(Deduction theorem) If* $\Gamma, \psi \vdash \chi$*, then* $\Gamma \vdash \psi \to \chi$.
2. *($\Box$-Monotonicity) If* $\Gamma \vdash [\mathbf{X} = \mathbf{x}]\psi$, $\vdash \psi \to \psi'$ *and* $[\mathbf{X} = \mathbf{x}]\psi' \in \mathcal{H}$*, then* $\Gamma \vdash [\mathbf{X} = \mathbf{x}]\psi'$.
3. *($\Diamond$-Monotonicity) If* $\Gamma \vdash \langle \mathbf{X} = \mathbf{x} \rangle\psi$, $\vdash \psi \to \psi'$ *and* $\langle \mathbf{X} = \mathbf{x} \rangle\psi' \in \mathcal{H}$*, then* $\Gamma \vdash \langle \mathbf{X} = \mathbf{x} \rangle\psi'$.
4. $\vdash ([\mathbf{X} = \mathbf{x}]\psi \,\&\, [\mathbf{X} = \mathbf{x}]\chi) \leftrightarrow [\mathbf{X} = \mathbf{x}](\psi \,\&\, \chi)$.
5. *(Replacement) Suppose* $\Gamma \vdash \theta \leftrightarrow \theta'$. *Then* $\Gamma \vdash \varphi \leftrightarrow \varphi[\theta'/\theta]$*, provided this is an* $\mathcal{H}$ *formula.*
6. $\vdash {\sim}[\mathbf{X} = \mathbf{x}]\psi \leftrightarrow \langle \mathbf{X} = \mathbf{x} \rangle {\sim}\psi$

7. $\vdash \sim\langle \mathbf{X} = \mathbf{x} \rangle \psi \leftrightarrow [\mathbf{X} = \mathbf{x}] \sim\psi$

8. $\vdash (\langle \mathbf{X} = \mathbf{x} \rangle \psi \sqcup \langle \mathbf{X} = \mathbf{x} \rangle \chi) \leftrightarrow \langle \mathbf{X} = \mathbf{x} \rangle (\psi \sqcup \chi)$.

9. $\vdash ([\mathbf{X} = \mathbf{x}]\psi \,\&\, \langle \mathbf{X} = \mathbf{x} \rangle \top) \to \langle \mathbf{X} = \mathbf{x} \rangle \psi$.

10. $\vdash ([\mathbf{X} = \mathbf{x}]\psi \,\&\, \langle \mathbf{X} = \mathbf{x} \rangle \chi) \to \langle \mathbf{X} = \mathbf{x} \rangle (\psi \,\&\, \chi)$.

11. $\vdash \langle \mathbf{X} = \mathbf{x} \rangle (\psi \,\&\, \chi) \to (\langle \mathbf{X} = \mathbf{x} \rangle \psi \,\&\, \langle \mathbf{X} = \mathbf{x} \rangle \chi)$.

Proof. 1) By induction on the length of a proof of $\Gamma, \psi \vdash \chi$.

Suppose χ is an axiom. Then $\vdash \chi$. By I0, $\vdash \chi \to (\psi \to \chi)$. Thus, by MP, $\Gamma \vdash \psi \to \chi$.

Suppose χ is in Γ. Then, since $\vdash \chi \to (\psi \to \chi)$ by I0, we have $\Gamma \vdash \psi \to \chi$ by MP.

Suppose χ is ψ. Then $\Gamma \vdash \psi \to \psi$ by I0.

Suppose the last rule used is MP; then χ follows from assumptions θ and $\theta \to \chi$, such that $\Gamma \cup \{\psi\} \vdash \theta$ and $\Gamma \cup \{\psi\} \vdash \theta \to \chi$ (via subderivations). By i.h., $\Gamma \vdash \psi \to \theta$ and $\Gamma \vdash \psi \to (\theta \to \chi)$. Since, by I0, $\Gamma \vdash (\psi \to \theta) \to [(\psi \to (\theta \to \chi)) \to (\psi \to \chi)]$, we get $\Gamma \vdash \psi \to \chi$ by two applications of MP.

Suppose the last rule used is NEC; then χ is of the form $[\mathbf{X} = \mathbf{x}]\varphi$, and $\vdash \varphi$. Using NEC, we obtain $\vdash [\mathbf{X} = \mathbf{x}]\varphi$. By I0, $\vdash [\mathbf{X} = \mathbf{x}]\varphi \to (\psi \to [\mathbf{X} = \mathbf{x}]\varphi)$, so, by MP, $\vdash \psi \to [\mathbf{X} = \mathbf{x}]\varphi$. *A fortiori*, then, $\Gamma \vdash \psi \to [\mathbf{X} = \mathbf{x}]\varphi$.

2) If $[\mathbf{X} = \mathbf{x}]\psi, [\mathbf{X} = \mathbf{x}]\psi' \in \mathcal{H}$, then ψ, ψ' are modal-free. Thus, from $\vdash \psi \to \psi'$, by NEC, we get $\vdash [\mathbf{X} = \mathbf{x}](\psi \to \psi')$. By I5 and MP, $\vdash [\mathbf{X} = \mathbf{x}]\psi \to [\mathbf{X} = \mathbf{x}]\psi'$. Thus, by MP, from the assumption that $\Gamma \vdash [\mathbf{X} = \mathbf{x}]\psi$ we obtain $\Gamma \vdash [\mathbf{X} = \mathbf{x}]\psi'$.

3) Remember that $\langle \mathbf{X} = \mathbf{x} \rangle$ abbreviates $\neg[\mathbf{X} = \mathbf{x}]\neg$. From $\vdash \psi \to \psi'$ we obtain $\vdash \sim\psi' \to \sim\psi$ by I0 and MP. As in the proof of point 2., we can show that $\sim\psi' \to \sim\psi$ is modal-free. Thus, by NEC and I5, $\vdash [\mathbf{X} = \mathbf{x}]\sim\psi' \to [\mathbf{X} = \mathbf{x}]\sim\psi$. Again by I0 and NEC, we obtain $\vdash \sim[\mathbf{X} = \mathbf{x}]\sim\psi \to \sim[\mathbf{X} = \mathbf{x}]\sim\psi'$. Thus, from $\Gamma \vdash \sim[\mathbf{X} = \mathbf{x}]\sim\psi$ we obtain $\Gamma \vdash \sim[\mathbf{X} = \mathbf{x}]\sim\psi'$ by MP.

4) Assume $[\mathbf{X} = \mathbf{x}]\psi \,\&\, [\mathbf{X} = \mathbf{x}]\chi$. We obtain $[\mathbf{X} = \mathbf{x}]\psi$ and $[\mathbf{X} = \mathbf{x}]\chi$ by I0 and MP. Since ψ and χ are modal-free, by I0 and NEC, we have $[\mathbf{X} = \mathbf{x}](\psi \to (\chi \to (\psi \,\&\, \chi)))$. Applying I5 and MP, we obtain $[\mathbf{X} = \mathbf{x}](\chi \to (\psi \,\&\, \chi))$. Again by I5 and MP, we obtain $[\mathbf{X} = \mathbf{x}](\psi \,\&\, \chi)$. Then $([\mathbf{X} = \mathbf{x}]\psi \,\&\, [\mathbf{X} = \mathbf{x}]\chi) \to [\mathbf{X} = \mathbf{x}](\psi \,\&\, \chi)$ by the deduction theorem.

In the opposite direction, assume $[\mathbf{X} = \mathbf{x}](\psi \& \chi)$. By I0 and NEC we have $[\mathbf{X} = \mathbf{x}]((\psi \& \chi) \to \psi)$. By I5, $[\mathbf{X} = \mathbf{x}](\psi \,\&\, \chi) \to [\mathbf{X} = \mathbf{x}]\psi$. We then obtain $[\mathbf{X} = \mathbf{x}]\psi$ by MP. $[\mathbf{X} = \mathbf{x}]\chi$ is obtained analogously.

5) Induction on φ.

If φ is an atom, then either $\theta = \varphi$ or θ is not a subformula of φ. Both cases are easy.

Case φ is $\psi \,\&\, \chi$. We prove this case by the deduction theorem. Assume Γ and φ; then, we get ψ and χ by I0+MP. By the i.h., $\Gamma \vdash \psi \leftrightarrow \psi[\theta'/\theta]$ and $\Gamma \vdash \chi \leftrightarrow \chi[\theta'/\theta]$. Thus, by MP, we get $\Gamma, \varphi \vdash \psi[\theta'/\theta]$ and $\Gamma, \varphi \vdash \chi[\theta'/\theta]$. By I0+MP, we conclude $\Gamma, \varphi \vdash \varphi[\theta'/\theta]$. So, by the deduction theorem, $\Gamma \vdash \varphi \to \varphi[\theta'/\theta]$. The converse direction is analogous.

Case φ is $\sim\psi$. We have $\Gamma \vdash \theta \leftrightarrow \theta'$; by the i.h., $\Gamma \vdash \psi \leftrightarrow \psi[\theta'/\theta]$. Since, by I0, $\Gamma \vdash (\psi \leftrightarrow \psi[\theta'/\theta]) \leftrightarrow (\sim\psi \leftrightarrow \sim\psi[\theta'/\theta])$, we conclude $\Gamma \vdash \sim\psi \leftrightarrow \sim\psi[\theta'/\theta]$.

Case φ is $[\mathbf{X} = \mathbf{x}]\psi$. By the i.h., $\Gamma \vdash \psi \leftrightarrow \psi[\theta'/\theta]$. By NEC, $\Gamma \vdash [\mathbf{X} = \mathbf{x}](\psi \leftrightarrow \psi[\theta'/\theta])$. By point 4., plus I0+MP, we obtain $\Gamma \vdash [\mathbf{X} = \mathbf{x}](\psi \to \psi[\theta'/\theta])$ and $\Gamma \vdash [\mathbf{X} = \mathbf{x}](\psi[\theta'/\theta] \to \psi)$. By axiom I5 and I0+MP, we obtain the desired conclusion.

6) We use once more the deduction theorem. Assume first $\sim[\mathbf{X} = \mathbf{x}]\psi$. We have $\vdash \psi \leftrightarrow \sim\sim\psi$ by I0, so by replacement and MP we obtain $\sim[\mathbf{X} = \mathbf{x}]\sim\sim\psi$. This is $\langle \mathbf{X} = \mathbf{x} \rangle \sim\psi$ by definition.

Vice versa, assume $\langle \mathbf{X} = \mathbf{x} \rangle \sim\psi$, i.e.$\sim[\mathbf{X} = \mathbf{x}]\sim\sim\psi$. By I0, replacement and MP, we obtain $\sim[\mathbf{X} = \mathbf{x}]\psi$.

7) $\sim\langle\mathbf{X}=\mathbf{x}\rangle\psi$ abbreviates $\sim\sim[\mathbf{X}=\mathbf{x}]\sim\psi$. By I0 and MP, it is equivalent to $[\mathbf{X}=\mathbf{x}]\sim\psi$.

8) Assume $\langle\mathbf{X}=\mathbf{x}\rangle\psi \sqcup \langle\mathbf{X}=\mathbf{x}\rangle\chi$; it abbreviates $\sim(\sim\langle\mathbf{X}=\mathbf{x}\rangle\psi \,\&\, \sim\langle\mathbf{X}=\mathbf{x}\rangle\chi)$. By part 7. and replacement (twice) we obtain $\sim([\mathbf{X}=\mathbf{x}]\sim\psi \,\&\, [\mathbf{X}=\mathbf{x}]\sim\chi)$. Then, by part 4. and MP, we obtain $\sim[\mathbf{X}=\mathbf{x}](\sim\psi \,\&\, \sim\chi)$. By part 6., then, $\langle\mathbf{X}=\mathbf{x}\rangle\sim(\sim\psi \,\&\, \sim\chi)$, i.e. $\langle\mathbf{X}=\mathbf{x}\rangle(\psi \sqcup \chi)$. The converse is similar.

10) From the negation of the conclusion we obtain $[\mathbf{X}=\mathbf{x}]\sim(\psi \,\&\, \chi)$ by point 7. Together with the assumption $[\mathbf{X}=\mathbf{x}]\psi$, by point 4., we obtain $[\mathbf{X}=\mathbf{x}](\sim(\psi \,\&\, \chi) \,\&\, \psi)$, and thus by I0 and replacement $[\mathbf{X}=\mathbf{x}]\sim\chi$. Finally, by point 7. again, $\sim\langle\mathbf{X}=\mathbf{x}\rangle\chi$.

9) By point 10. we have $\vdash ([\mathbf{X}=\mathbf{x}]\psi \,\&\, \langle\mathbf{X}=\mathbf{x}\rangle\top) \to \langle\mathbf{X}=\mathbf{x}\rangle(\psi \,\&\, \top)$. By I0 and replacement, $\langle\mathbf{X}=\mathbf{x}\rangle(\psi \,\&\, \top) \to \langle\mathbf{X}=\mathbf{x}\rangle\psi$. Thus, by I0 and point 2., $\vdash ([\mathbf{X}=\mathbf{x}]\psi \,\&\, \langle\mathbf{X}=\mathbf{x}\rangle\top) \to \langle\mathbf{X}=\mathbf{x}\rangle\psi$.

11) Assume $\langle\mathbf{X}=\mathbf{x}\rangle(\psi \,\&\, \chi)$. Since $\vdash (\psi \,\&\, \chi) \to \psi$, by monotonicity (point 3.) we obtain $\Diamond\psi$. We obtain analogously $\Diamond\chi$. Thus, $\Diamond\psi \,\&\, \Diamond\chi$ by I0. □

B - Maximally consistent sets of formulas

The following proofs are routine, and included for reference.

Lemma B.1. *Let $\Gamma \cup \{\chi\} \subseteq \mathcal{H}$. If $\Gamma \vdash \chi$ and $\Gamma \vdash \sim\chi$, then $\Gamma \vdash \bot$.*

Proof. Remembering that $\bot$ abbreviates $X = x \,\&\, \sim X = x$, We observe that $\chi \to (\sim\chi \to \bot)$ is an instance of a tautology, so we obtain $\Gamma \vdash \bot$ by I0 and two applications of MP. □

Lemma B.2. *Suppose $\Gamma \nvdash \chi$. Then, $\Gamma \cup \{\sim\chi\}$ is consistent.*

Proof. Suppose it is not; then $\Gamma \cup \{\sim\chi\} \vdash \psi, \sim\psi$ for some formula ψ. By I0, $\vdash \psi \to (\sim\psi \to \psi \,\&\, \sim\psi)$, so by MP, $\Gamma \cup \{\sim\chi\} \vdash \psi \,\&\, \sim\psi$. By the deduction theorem, $\Gamma \vdash \sim\chi \to (\psi \,\&\, \sim\psi)$. By I0 and MP, we obtain $\Gamma \vdash \chi$, contrarily to the assumption. □

Lemma 4.5. *Let Γ be a maximally consistent set of $\mathcal{H}_\sigma$ formulas (or $\mathcal{H}^+_\sigma$ formulas). Then:*

1. *Closure under $\vdash$: if $\Gamma \vdash \psi$, then $\psi \in \Gamma$.*
2. *Completeness: for every formula ψ, either ψ or $\sim\psi$ is in Γ.*
3. *Closure under &: if $\psi, \chi \in \Gamma$, then $\psi \,\&\, \chi \in \Gamma$.*
4. *Primality: if $\psi \sqcup \chi \in \Gamma$, then $\psi \in \Gamma$ or $\chi \in \Gamma$.*
5. *Impossibility of contradiction: $\Diamond\bot \notin \Gamma$.*

Proof. 1) Suppose $\Gamma \vdash \psi$. Then, Γ and $\Gamma \cup \{\psi\}$ have the same set of consequences; thus, by the maximality of Γ, $\psi \in \Gamma$.

2) Suppose first that $\Gamma \vdash \psi$. Then, $\psi \in \Gamma$ by point 1.

If instead $\Gamma \nvdash \psi$, then, by lemma B.2, $\Gamma \cup \{\sim\psi\}$ is consistent, and $\Gamma \cup \{\sim\psi\} \supsetneq \Gamma$; thus, by the maximality of Γ, $\sim\psi \in \Gamma$.

3) By the assumptions, $\Gamma \vdash \psi, \chi$. By I0, $\Gamma \vdash \psi \to (\chi \to (\psi \,\&\, \chi))$. Thus, by two applications of MP, $\Gamma \vdash \psi \,\&\, \chi$. Thus, by 1., $\psi \,\&\, \chi \in \Gamma$.

4) Suppose $\psi \notin \Gamma$ and $\chi \notin \Gamma$. By 2., then, $\sim\psi \in \Gamma$ and $\sim\chi \in \Gamma$. By 3., $\sim\psi \,\&\, \sim\chi \in \Gamma$. By I0 and part 2., $\sim\sim(\sim\psi \,\&\, \sim\chi) \in \Gamma$, i.e. $\sim(\psi \sqcup \chi) \in \Gamma$. By the consistency of Γ, then, $\psi \sqcup \chi \notin \Gamma$.

5) Note that $\Diamond\bot$ abbreviates $\sim\Box\sim\bot$, i.e., $\sim\Box\top$. On the other hand, $\Box\top \in \Gamma$ by I0 and NEC. Thus, by the consistency of Γ, $\sim\Box\top \notin \Gamma$. □